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Scaling relations for forced oscillators at the transition from a dissipative to a Hamiltonian system

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The dynamics and the stability of a forced damped nonlinear oscillator driven at twice its resonance frequency are studied. At the transition from a dissipative system to a Hamiltonian system, simple scalings relations are found by the use of the Floquet theory of the linearized problem. The Floquet exponents and the period-doubling bifurcation point are determined analytically in the limit of small damping. The theory is compared to numerical calculations on a Duffing oscillator and excellent agreement is found.

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Nonlinear oscillators appear in many different contexts in physics, chemistry, and biology. Therefore a deeper understanding of universal features for classes of oscillators is important. Most oscillators are coupled to an environment and are as a result damped. In some cases this damping is small and can be neglected, and the resulting "model" system is said to be Hamiltonian. But for some features of a nonlinear oscillator the transition between a dissipative and a Hamiltonian systems can be subtle and as a result the dynamics cannot be derived from a Hamiltonian system no matter how small the dissipation is. The existence of universal features at the transition between dissipative and Hamiltonian systems is well known and has particularly been studied for the transition to chaos through an infinite period-doubling sequence [1, 2]. The present paper deals also with universality for a system that is able to undergo a period-doubling bifurcation, but instead of looking at the infinite sequence, we study the features of a oscillator at the first period-doubling bifurcation.

The paper is organized as follows. First a nonlinear damped oscillator is introduced. This equation is then linearized around a stable limit cycle with period T. From the linearized equation the stability of the limit cycle is determined by the use of Floquet theory. In particular the bifurcation point as a function of the system parameters is determined. This result will be used to rescale the amplitude of the driving field and to derive scaling relations. We then discuss the response of the system to a perturbation which also will exhibit universal features. Finally the theoretical results of the paper are compared with numerical integration of the forced Duffings equation. In particular, the numerically determined bifurcation point is compared with the theory.

The forced damped nonlinear oscillator is assumed to be of the type

$$x_{tt} + \alpha x_t + \frac{dV(x)}{dx} = A_D \cos(\omega_D t), \tag{1}$$

where α is a damping constant and $V(x)$ is a nonlinear potential, which for small x, linearized around a minimum x_m, has the Taylor expansion $V(x) = \frac{1}{2}\omega_0^2 x^2 + \frac{1}{6}\gamma x^3 + \cdots$, where ω_0 and γ are nonzero. Finally A_D and ω_D are the amplitude and the frequency of the drive of the driving field. Furthermore, it is assumed that the particle does not jump to another minimum.

The above equation can be linearized about the periodic limit cycle $x_0(t)$. This gives, due to the periodic coefficients, a Floquet equation

$$\xi_{tt} + \alpha \xi_t + \left.\frac{d^2V}{dx^2}\right|_{x_0} \xi = 0. \tag{2}$$

From this linear equation the stability of the limit cycle can in principle be found, but the solution (limit cycle x_0) to the full nonlinear problem has to be known. Even if the full solution is known it is in general not possible to find a closed expression for the stability as a function of the system parameters [3]. The present study is, however, an exception where the limit cycle and its stability can be found analytically. This occurs at the transition from dissipative to a Hamiltonian system, when the system is driven at twice its resonance frequency and in the restricted parameter space where the amplitude of the driving field is smaller than its critical value.

The periodic coefficient in Eq. (1) can be written

$$\left.\frac{d^2V}{dx^2}\right|_{x_0} = \omega_0^2 + \gamma x_0(t) + \mathcal{O}(x_0^2). \tag{3}$$

There exist two solutions to the above equation of the form [3, 4]

$$\xi_\pm = e^{\rho_\pm t} P_\pm(t), \quad \xi_\pm(t + 2\pi/\omega_D) = \mu_\pm P_\pm(t), \tag{4}$$

where $\mu_\pm = \exp(2\pi\rho_\pm/\omega_D)$ are called the Floquet multipliers and are the $\rho_\pm$ Floquet exponents. The Floquet multipliers play a very important role here. The function $P_\pm$ is periodic with the same period as the solution, x_0, to the nonlinear problem

$$P_\pm(t + 2\pi/\omega_D) = P_\pm(t). \tag{5}$$

The two Floquet multipliers are constrained by the dissipation in the system through

$$\mu_1\mu_2 = \exp(-\kappa), \tag{6}$$

where $\kappa = 2\pi\alpha/\omega_D$. As the bifurcation point is approached one of the multipliers tends to -1, and the other to $\exp(-2\pi\alpha/\omega_D)$.

Next we argue that the bifurcation point A_C scales to zero when the damping tends to zero. This is done by taking the limit $A_D \to 0$ followed by $\alpha \to 0$. Now the linearized equation (2) is simply that of a harmonic oscillator. However, in this case the system is only marginally stable against a period-doubling bifurcation ($\mu_1 = \mu_2 = -1$)[5]. In other words the bifurcation point has scaled to zero amplitude for a driving field at twice the resonant frequency. If the oscillator is not driven at twice the resonant frequency a finite threshold amplitude is needed in order for the system to bifurcate. This will become clear below where the Floquet exponents are determined.

Knowing that the critical value of the driving field scales to zero it is expected that, as the damping tends to zero, there exist a scaling relation between A_C and α. Since the bifurcation point scales to zero so does the limit cycle x_0, which means that the limit cycle can be found by linear response. In this limit, the limit cycle x_0 is given by $x_0 = (A/3)\omega_0^2\cos(2\omega_0 t)$, which we find by solving Eq. (1) for $x \ll 1$ and $\alpha \ll 1$.

We now determine the Floquet exponents. By removing the damping term in Eq. (2) with the transformation $\xi' = \xi\exp(\alpha t/2)$ and subsequently Fourier transforming, we obtain

$$(-\omega^2 n^2 + 2i\omega n\rho + \omega_R^2 + \rho^2)p_n + \frac{\epsilon}{2}(p_{n+1} + p_{n-1}) = 0, \tag{7}$$

where p_n is the Fourier transform of $P_\pm$, $\omega_R^2 = \omega_0^2 - (\alpha/2)^2$, and $\epsilon = \gamma A/3\omega_R{}^2$. This is a trigonal matrix equation which can be solved along the same lines of reasoning as for Bloch states in a weak periodic potential. For $\epsilon = 0$, we have the solution $\rho = \pm i\omega_R$, and $p_n = 0$ unless $n = 0$. For $\epsilon \neq 0$, p_n is different from zero for all n, but since they will be of order ϵ, it is straightforward to see that the only p_n's which are nonzero to linear order in ϵ are $p_{\pm 1}$. We can thus truncate Eq. (7) at $n = \pm 1$, and we have

$$\begin{pmatrix} H_1 & \epsilon/2 & 0 \\ \epsilon/2 & H_0 & \epsilon/2 \\ 0 & \epsilon/2 & H_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_0 \\ p_{-1} \end{pmatrix} = 0, \tag{8}$$

where $H_n = -\omega^2 n^2 + 2i\omega n\rho + \omega_R^2 + \rho^2$. The Floquet exponents, $\rho_\pm$, are found by requiring that the determinant of the matrix above vanish. If we use that $R \equiv \rho^2 + \omega_R^2$ is of the order ϵ we obtain

$$R = \Delta^2 - \sqrt{\Delta^2 - \left(\frac{\epsilon}{2}\right)^2}, \tag{9}$$

where $\Delta = \sqrt{\omega_R^2 - (\omega_D/2)^2}$ is the detuning away from the resonance for half-harmonic generation. This finally gives us the Floquet exponents:

$$\rho_\pm \approx i\left(\omega_R \mp \frac{R}{2\omega_R}\right). \tag{10}$$

Note the nonanalytic dependence of the driving field, ϵ, which is a consequence of the self-consistent calculation of the Floquet exponents. If the system is driven away from resonance and if ϵ is sufficiently smaller than the detuning, we see that R remains real and will just shift the frequency by a small amount. At a critical value R gets an imaginary part and the system is driven toward the instability. This has a simple geometrical interpretation in terms of the Floquet multipliers discussed in Ref. [6]. The situation when the square root in Eq. (9) is zero translates exactly to the point where the Floquet multipliers meet at the real axis.

The stability of the solution is determined from $\rho_\pm$. The critical value is determined by $\mathrm{Re}[\rho_+] = \alpha/2$. Using this criteria, the amplitude of the driving field at the bifurcation point becomes, in the special case where $\Delta = 0$,

$$A_C = \frac{6\alpha\omega_R^3}{\gamma} = \frac{6\alpha[V''(x_m)]^{3/2}}{V'''(x_m)}. \tag{11}$$

This is a useful result since the expression for A_C can be used to scale the amplitude of the driving field. This agrees with a result found by Pedersen, Samuelsen, and Saermark [7], who considered the special case of the resisted shunted Josephson junction (RSJ) model. It should, however, be emphasized that the present formula for the bifurcation point is a general result.

The two solutions of the homogeneous equation are now found to be given by

$$\xi_\pm(t') = [\cos(\omega_R' t') \mp \sin(\omega_R' t')]\exp\left[-\left(1 \pm \frac{A}{A_C}\right)t'\right]. \tag{12}$$

Here we have rescale time and frequency: $t' = t\alpha/2$ and $\omega_R' = \omega_R 2/\alpha$. The functions are in fact *universal* in the sense that for a large class of nonlinear potentials the deviations from the limit cycle have this form. The class is defined by having nonzero ω_0 and γ. It is seen from the expression for A_C that the bifurcation point moves to infinity for vanishing γ. This agrees with the experience that the limit cycle has to be nonsymmetric in time (contain even harmonics in its Fourier spectrum), since the γ term breaks the symmetry [8]. In summary, we have, in the limit of small damping, found the explicit form of the solutions to the homogeneous Mathieu equation and found a formula for the critical value of the driving field. We also saw that the non-linearity in the limit of a small driving amplitude does not alter the periodic part of the solutions which are simply given by that of a driven harmonic oscillator. The nonlinearity enters only in the exponents which determine the stability of the solutions. The key to this was the use of Floquet theory which enabled us in a nonperturbative fashion to find these exponents.

From the above solution to the homogeneous equation, solutions to small perturbations on Eq. (1) can be calculated. These solutions will, when scaled with the damping, also possess the same scaling relations as found for the homogeneous equation. An example is the squeezing spectrum of thermal fluctuations as studied in Ref. [6, 9]. However, since the response diverges at the bifurcation point, the scaling relations for the response are not valid in the vicinity of the bifurcation point.

We have compared the simple law for the bifurcation point with numerical simulations of a driven Duffing's oscillator. The governing differential equation is given by

$$x_{tt} + \alpha x_t + x + x^3 = A_D \cos(\omega_D t) + \eta, \tag{13}$$

where α is the damping coefficient, A_D and ω_D are the amplitude of the driving field and frequency, respectively. Finally, η is a constant force term. This equation was solved by a fourth-order Runge-Kutta method. In all the calculations the frequency of the drive is fixed at twice the resonance frequency of the system. The equation was solved for increasing values of the driving amplitude. For each value of the drive the solution was converged towards the limit cycle so that the difference between to cycles $x_0(2\pi/\omega_D + t) - x_0(t)$ was less than 10^{-7}. This limit cycle was used to determine the two Floquet multipliers numerically. The accuracy of the method was checked by making sure that the product of the two multipliers $\mu_1\mu_2$ was within 10^{-5} of the theoretical value $\exp(-\alpha 2\pi/\omega_D)$. The bifurcation point was then identified as the value of A_D where one of the multipliers was equal to -1.

In Fig. 1 the scaled bifurcation point A_C/α as a function of η is plotted for two different values α. The solid curve is the theoretical determined value of A_C/α as a function of η, and is in this case given by

$$\frac{A_C}{\alpha} = \frac{(1 + 3x_m^2)^{\frac{3}{2}}}{x_m}, \tag{14}$$

where

$$x_m = \sqrt[3]{-\frac{\eta}{2} + \sqrt{\frac{1}{27} + \frac{\eta^2}{4}}} - \sqrt[3]{\frac{\eta}{2} + \sqrt{\frac{1}{27} + \frac{\eta^2}{4}}}, \tag{15}$$

and here x_m is the minimum of the potential. There is seen to be a very good agreement between the theory and the numerically found bifurcation point. For the higher damping there is a small deviation from the theoretical curve, but as the damping becomes smaller the numerical determined points reproduce the theoretical curve. We have also performed numerical calculations for the forced pendulum (or the RSJ model of a Josephson junction) and also in this case we find good agreement with the theory.

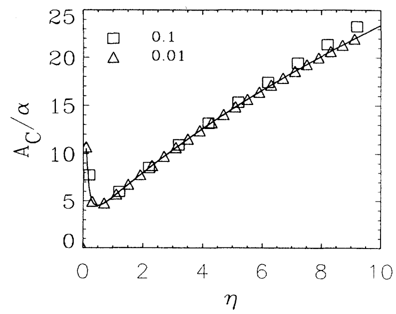

FIG. 1. The scaled bifurcation point A_C/α for the Duffing's oscillator as a function of η. The points are numerically determined and the solid curve is the theory. The numerically used values of α are 0.1 and 0.01. The figure shows that the theory accounts well for the numerical results even for the larger value of the damping. We also see that even though the scaling relation between α and A_C is simple the relation between A_C and the remaining system parameters is in general complicated.

In conclusion a nonlinear oscillator driven at twice its resonance frequency has been studied. The emphasis has been on the dynamics at the transition between dissipative and a Hamiltonian system. In this limit the period-doubling bifurcation point A_D has been found by using Floquet theory on the linearized equation. The bifurcation point scaled in a simple way with the damping but may at the same time be a complicated function of the remaining system parameters. Rescaling the solution with the determined value of A_C gives a solution with universal features. The response of the system to a small perturbation will have the same universal features which follows from the fact that the response to a perturbation is constructed from the homogeneous solution. The theory was compared with numerically calculations on the driven damped nonlinear Duffing's oscillator and a remarkably good agreement was found. The numerical results were found for the system driven at twice its resonant frequency. However, the theory should also work for a finite detuning; see Eq. (9). We note that our results are only valid at the transition to the corresponding Hamiltonian system but as the numerical example shows, the features we found at this fixed point are indeed expected to describe the system correctly for finite values of the damping as well. It should be possible to investigate the features of the present theory experimentally.

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